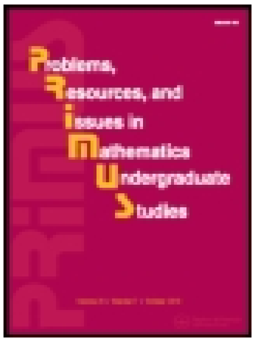

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Inquiry-Oriented Instruction: A Conceptualization of the Instructional Principles

George Kuster, Estrella Johnson, Karen Keene, and Christine Andrews-Larson

Abstract: Research has highlighted that inquiry-based learning (IBL) instruction leads to many positive student outcomes in undergraduate mathematics. Although this research points to the value of IBL instruction, the practices of IBL instructors are not well-understood. Here, we offer a characterization of a particular form of IBL instruction: inquiry-oriented instruction. This characterization draws on K-16 research literature in order to explicate the instructional principles central to inquiry-oriented instruction. As a result, this conceptualization of inquiry-oriented instruction makes connections across research communities and provides a characterization that is not limited to undergraduate, secondary, or elementary mathematics education.

Keywords: Inquiry-oriented, instructional practices, teaching.

1. INTRODUCTION

Recently published empirical studies provide evidence that inquiry-based learning (IBL) has many positive outcomes in the teaching of undergraduate mathematics in the United States. It has been reported that:

> students in IBL math-track courses reported greater learning gains than their non-IBL peers on every measure: cognitive gains in understanding and thinking; affective gains in confidence, persistence, and positive attitude about mathematics; and collaborative gains in working with others, seeking help, and appreciating different perspectives [22, p. 409].

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Color versions of one or more of the figures in the article can be found online at www.tandfonline.com/upri.

In addition to these positive outcomes for all students, IBL also functioned as a more equitable form of instruction, as evidenced by the elimination of gender differences in student outcomes. More specifically:

> women in non-IBL courses reported substantially lower cognitive and affective gains than did their male classmates. In contrast, in IBL courses, women's cognitive and affective gains were statistically identical to those of men, and their collaborative gains were higher [22, p. 412].

Further supporting these findings is a previous study by Kogan and Laursen [17], in which the authors analyzed data from over 100 sections of 40 courses, and found that students who had taken an IBL course were more likely to pursue further math courses, and that these students were just as likely as their non-IBL peers to succeed in subsequent courses (even though less material may have been covered in the IBL courses).

Generally speaking, IBL instruction has been described as a "big tent" that encompasses many different instructional approaches. However, in all of these approaches there is a commitment to active learning. As described by Laursen et al. [22]:

> IBL methods invite students to work out ill-structured but meaningful problems …… Following a carefully designed sequence of tasks rather than a textbook, students construct, analyze, and critique mathematical arguments. Their ideas and explanations define and drive progress through the curriculum. In class, students present and discuss solutions alone at the board or via structured small-group work while instructors guide and monitor this process [22, p. 407].

Whereas this description provides an account of what students are doing in IBL classes, we hope to further the discussion by elaborating on what teachers do to facilitate this sort of student engagement.

As we do not speak for the entire IBL community, our discussion will instead focus on providing a framework for *inquiry-oriented instruction*: a form of instruction that shares many instructional philosophies and goals with IBL. Specifically, we draw on our collective research experience with *inquiry-oriented instruction* [12, 13, 14, 15, 18, 19, 20, 21, 34, 36, 37], situated within the theory of Realistic Mathematics Education [5], to present a characterization of *inquiry-oriented instruction* framed around four instructional principles: *generating student ways of reasoning, building on student contributions, developing a shared understanding*, and *connecting to standard mathematical language and notation*. Situating this form of instruction with regard to IBL, we believe that some distinguishing characteristics of discussions surrounding inquiry-oriented instruction include their heavy emphasis on student reinvention of concepts, instructor inquiry into student thinking, and student inscriptions and their role in the development of the mathematics.

To better understand these principles and their nuances, and describe each of their roles in instruction, we first draw on the Realistic Mathematics Education theory and research literature (especially that research carried out in undergraduate mathematics). We then look to the broader K-16 literature base to understand, and illustrate, how these principles can be enacted. In the following sections we present a general overview of inquiry-oriented instruction and its philosophical foundations, and then draw connections with the four instructional principals. After providing a description of each of the principles and what they may look like in action, we conclude the paper with a vignette from an introductory linear algebra class and a brief discussion.

2. INQUIRY-ORIENTED INSTRUCTION

The framing of inquiry-oriented instruction we provide draws on the work underlying three sets of research-based, inquiry-oriented curricular materials currently being scaled up for post-calculus undergraduate mathematics courses: linear algebra (e.g. [35]), differential equations [25], and abstract algebra [21]. Each set of these curricular materials was designed in accordance with the instructional design theory of Realistic Mathematics Education and is intended to foster student reinvention of mathematical concepts [4]. To support this reinvention, the curricular materials include task sequences "that build on student concepts and reasoning as the starting point from which more complex and formal reasoning develops" [35, p. 321]. These task sequences form the basis of inquiry-oriented instruction, by providing a trajectory for the students' reinvention process, a context for student inquiry, and a foundation on which teachers can access, inquire into, and guide student understanding.

Students in an inquiry-oriented classroom:

> learn new mathematics through *inquiry* by engaging in mathematical discussions, posing and following up on conjectures, explaining and justifying their thinking, and solving novel problems [27, p. 190].

These activities promote the emergence of important student-generated ideas and solution methods, which one can think of as the mathematical "fodder" available to the teacher for the progression of the mathematical agenda [31]. This fodder is generated by the students as they engage with the mathematical activities that comprise the instructional sequence, and by participating in argumentation and justification as students explain their own ways of reasoning and make sense of the reasoning of others. In this way, student inquiry and carefully designed task sequences provide instructional opportunities; however, it is with the support of the instructor that these opportunities are fostered, refined, and leveraged. As such, we follow Rasmussen and Kwon's [27] emphasis that, in addition to students' inquiry into mathematical problems, instructors' inquiry into student thinking is a crucial aspect of inquiry-oriented instruction.

3. INSTRUCTIONAL PRINCIPLES OF INQUIRY-ORIENTED INSTRUCTION

Reflecting on the research and descriptions of inquiry-oriented instruction, our own prior research on the implementation of inquiry-oriented instructional materials [12, 13, 14, 16, 18, 19, 20, 21, 34, 36, 37], and a collection of classroom videos of experts and novices implementing inquiry-oriented instructional materials [18], we were able to distill four instructional principles. With inquiry-oriented instruction, (informed by Realistic Mathematics Education) one of the main instructional tenets is that students' informal mathematical reasoning is evoked and then leveraged to develop the more formal mathematics [4]. This tenet gives rise to the first two instructional principles: *generating student ways of reasoning* and *building on student contributions*. Given the central role of students' mathematical reasoning in the development of the mathematical agenda, this reasoning must be generated, elicited, and inquired into by the teacher. The teacher then must help transition the developing mathematics of the classroom, from that of informal student reasoning to that of formal mathematics. To do so the teacher uses the students' contributions to build towards the mathematical agenda, while ensuring that the learners "come to regard the knowledge they acquire as their own private knowledge, knowledge for which they themselves are responsible" [6].

As the teacher is working to facilitate the generation of student mathematical reasoning, and even when the teacher is building on student contributions to advance the mathematical agenda, it may be the case that the mathematics of only a few individual students is the focus. For instance, it could be that only a handful of students have contributed to the whole-class discussion or that only one or two groups of students have developed the solution methods the teacher had intended. To help ensure that each student in the class constructs an understanding of the intended mathematics, the teacher must make explicit efforts to *develop a shared understanding* within the classroom community. This principle, reflected in the literature in terms of the teacher acting as a broker between the classroom community and individual groups of students within the classroom [29], and in terms of the teacher supporting classroom mathematical practices that become "taken-as-shared" within the classroom [33], reflects a commitment to considering the learning of the classroom collective as opposed to the learning of the few (often very strong) students contributing most frequently to whole-class discussions.

The final principle, *connecting to standard mathematical language and notation*, reflects the importance of the teacher acting as a broker between the classroom and the broader mathematics community. As described by [29]: "the teacher as broker might introduce formal terminology from the discipline of mathematics into the classroom community" (p. 174–175). This principle is inherently tied to the Realistic Mathematics Education ideal that "formal mathematics comes to the fore as a natural extension of the student's experiential reality" [5]. Thus, in an inquiry-oriented classroom, the development of formal mathematics is still the instructional goal; however, it is a consequence of (and not a starting point for) students' mathematical reasoning.

In the following subsections, we continue to draw on literature concerning inquiry-oriented instruction and Realistic Mathematics Education to more fully explicate the four inquiry-oriented instructional principles: (i) generating student ways of reasoning; (ii) building on student contributions; (iii) developing a shared understanding; and (iv) connecting to standard mathematical language and notation. We then look to the more extensive K-16 literature base to understand and illustrate how these principles can be enacted. It should be noted that the four principles are somewhat artificially separated for the purposes of explication. In actuality, these principles are quite intertwined and work together in different ways throughout the lesson to support various aspects of instruction.

3.1. Generating student ways of reasoning

In inquiry-oriented instruction, teachers engage the students in authentic mathematical activity, usually through the use of purposefully designed tasks that lead the students to discover key mathematical ideas [20, 28, 30, 31]. This authentic mathematical activity can be characterized as "complex thinking and reasoning strategies that would be typical of 'doing mathematics'" [7, p. 529] and includes conjecturing, justifying, and defining [8, 11, 31]. This stands in contrast with more procedural and/or memorization-based activity.

Once these rich ways of reasoning are generated, either for individual students or within small groups, they need to become part of the classroom discourse, and thus an object for subsequent mathematical activity [23]. To do this, teachers must prompt students to explain their thinking and justify their solution strategies, with a focus on the reasoning the students utilized during the task as opposed to only the procedures used. Research on instructional quality indicates that the type of contributions teachers elicit is directly related to the students' opportunities to learn. Thus, it is important that teachers elicit thinking and reasoning that "uncover the mathematical thinking behind the answers" [10, p. 92]. The K-16 research literature discusses a number of ways in which teachers can promote and elicit meaningful student contributions through questioning: questioning can drive student investigation of mathematics, can support students in explaining their solution strategies, and can help the instructor understand students' thinking [24].

This last role of questioning, helping the teacher understand students' thinking, has been identified by Rasmussen and Kwon [27] as a defining characteristic of inquiry-oriented instruction. The questions asked by teachers not only direct student investigations and provide the teacher with insight into student thinking, but also they also help students refine and reflect on their own thought process [1, 9, 27]. In this way, by inquiring into student thinking, teachers are able to support students in generating more sophisticated ways of reasoning.

3.1.1. Description of generating student ways of reasoning in action

Students are engaged in generalizing their thinking to make mathematical claims that they test and, then either support or reject these claims with arguments, examples, and counterexamples. The teacher tries to connect the task to the students' prior experiences in ways that facilitate their continued and meaningful engagement in the task.

The teacher explicitly asks students to share their approaches to the tasks and the reasoning the students used to complete those tasks. The focus is not on what answers the students attain, but rather on how the students arrived at their particular answer, how they were thinking about the problem, the method they used, and justifications for choosing that method.

The teacher engages in conversations with the students about their thinking. These conversations indicate that the teacher is attempting to assess their own understanding of students' thinking. There are instances of probing, rephrasing of student contributions, and re-presentations of student reasoning. It seems as though the teacher tries to figure out "how are my students thinking about this?" as opposed to only evaluating answers for correctness.

3.2. Building on student contributions

To support the students' own reinvention of the mathematical ideas, rather than strictly adhering to a rigid set of lecture notes, "classroom participants (teacher and students) lay down a mathematical path as they go" [38, p. 117]. In this way, the teacher facilitates the students' construction of intended mathematical goals, while simultaneously allowing the students to retain ownership of the mathematics: a key tenet of Realistic Mathematics Education [6]. To co-construct this path, teachers elicit and inquire into student contributions to determine which ideas (correct or incorrect) are important and relevant to the development of the mathematics, and which ideas can be leveraged to build student thinking that is commensurate with the mathematical goals of the lesson [14, 21, 31]. In addition to eliciting and inquiring into student thinking, the teacher must also use these contributions to inform the lesson. Here we provide more detail about how eliciting, inquiring, and utilizing student contributions allow teachers to build on student contributions as they advance the mathematical agenda.

Leatham et al. [23] state that foundational to building on student thinking is, first, the presence of student mathematical thinking that "is likely to advance students' development of important mathematical ideas—whether the student thinking is mathematically significant" [23, p. 92]. After eliciting the contributions, the teacher must then assess the contributions' potential for productivity with regard to the development of the mathematical agenda. In this sense, the teacher is again inquiring into student mathematics, but now for the purposes of determining if and how student contributions can be utilized to promote a more sophisticated understanding of the mathematics. That is, the teacher is considering the contributions of one or more students in reference to the intended key mathematical ideas of the lesson, and how the contributions could be used to build toward those key ideas. Although teacher inquiry serves many functions and roles throughout a lesson (see [10, 13, 27]), with regard to building on student contributions, teacher inquiry allows teachers to form models of student thinking and understanding, reconsider important mathematical ideas in light of these models, and formulate questions and tasks that enable the students to build on those ideas [27]. This aligns with the notion that:

> an important part of mathematics teaching is responding to student activity, listening to student activity, notating student activity, learning from student activity, and so on [28, p. 414].

By eliciting and inquiring, the teacher can generate instructional space where:

> the nature of student mathematical thinking might compel one to take a particular path because of the opportunity it provides at that moment to build on that thinking to further student mathematical understanding [21, p. 118].

In other words, "instead of controlling the exact content that gets stated in a lecture, the teacher's responsibility is to monitor, select, and sequence student ideas" [12, p. 13] in ways that build toward the class's development of the intended mathematics. Some of the ways a teacher can guide the development of the mathematical agenda by using student contributions include: identifying and sequencing student solutions to "ensure that the discussion advances his or her instructional agenda" [12, p. 648]; utilizing Pedagogical Content Tools "to connect to student thinking while moving the mathematical agenda forward" [28, p. 389]; or by refocusing the class towards the use of certain student-generated ideas, marking important student contributions, and assigning tasks meant to clarify and build on students' ideas/questions. In these ways, teachers can guide and manage the development of the lesson while building on student contributions, developing mathematical ideas in directions commensurate with the mathematical agenda, and supporting students in maintaining authority and ownership of the mathematics.

3.2.1. Description of building on student contributions in action

The teacher listens to and explores students' contributions when appropriate, and utilizes these ideas to inform follow-up questions and tasks that build along a trajectory toward the important key mathematical ideas. The teacher uses intermediate student progress to redirect and focus small-group work towards the important mathematical ideas. The instructor often takes advantage of student thinking that he/she feels can promote the emergence of ideas commensurate with the mathematical agenda. There is an atmosphere of openness to, and space for, following/exploring unexpected contributions; specifically, where the students are the ones doing the exploration. For instance, the teacher (knowing the mathematics that may develop as a consequence) may prompt a group of students to think about a particular approach, one a different group of students had found useful.

The teacher monitors small-group work and both records and uses student contributions during whole-class discussions as a way to promote the development of the intended mathematical ideas. The teacher guides student explorations towards the mathematical goals of the day. The teacher's questions and interactions appear to be directed towards a clear mathematical goal. Additionally, the instructor manages both the small-group work and the whole-class discussions. For example, the teacher may redirect the classroom's focus from small-group work to a whole-class discussion when productive ideas, which may help build on the ideas of other students, have emerged in some of the small groups.

3.3. Developing a shared understanding

As discussed by Stein et al. [32]:

> a key challenge that mathematics teachers face in enacting current reforms is to orchestrate whole-class discussions that use students' responses to instructional tasks in ways that advance the *mathematical learning of the whole class* [32, p. 312, emphasis added].

Within the inquiry-oriented research base, many articles make use of, and highlight the importance of, developing a shared understanding (e.g., [26, 29, 33]). For instance, Stephan and Rasmussen [33] discuss ways in which important mathematical ideas and ways of reasoning, emerging from ideas originating with individual students or small groups of students, become *taken-as-shared* within a classroom. The important distinction between *building on student contributions* and *developing a shared understanding* is characterized by who is making sense of the evolving mathematical agenda. In the former, it is the teacher and a select group of students who have provided the bulk of the contributions; in the latter, it is the classroom community as a whole, developing and co-constructing a taken-as-shared understanding. As described by Fredericks:

> There is this risk that you can pose the problem and then you can have five groups share how they did it and then you can go to the next problem [without any additional discussion of the groups' ideas]. And you can assume that the students will make the connections, and some of them will and some of them won't. I think to really be effective you have to push yourself further than that. That you have to think about what those connections are and you have to make sure that they explicitly come out. Otherwise you don't know who got it and who didn't. You are right back to where you were when you taught the old way [13, pp. 13–14].

In order to develop a shared understanding of the emerging mathematics, teachers can again be responsive to student thinking, engage students in one another's thinking, and introduce language and notation that helps students to "get on the same page."

Just as with *building on student thinking*, by being responsive to student thinking and contributions, teachers can create new instructional space [14]. With regard to this principle, the instructional space is created for the purpose of developing a shared understanding within the classroom community. For instance, a teacher may recognize that the whole class could benefit from one group's problem-solving method, and in response, create a task in order for the particular method to become familiar to, and understood by, the other students in that class. In this way, the new task creates space for the class to develop a shared understanding of an important mathematical idea that may have originated from an individual student.

By creating such a task, the teacher is also asking students to engage with one another's thinking. By having students engage in one another's thinking, all students are presented with opportunities to deepen their own thinking, generate new ideas, and make mathematical connections. As discussed by Jackson et al. [11]: "the teacher plays a crucial role in mediating the communication between students to help them understand each other's explanations" [11, p. 648]. Stein et al. [32] provide several examples of how teachers can support students in making mathematical connections between differing student contributions and important mathematical ideas. Some of these examples include asking students to reflect on the contributions of other students, assisting students in drawing connections between the mathematics present in solution strategies and the various representations that may be utilized, and facilitating mathematical discussions about different student approaches for solving a particular problem. Doing so can prompt students to reflect on other students' ideas while evaluating and revising their own [2, 3].

Finally, teachers can strategically use the introduction of language and notation as a way to facilitate a shared understanding. Mathematical language and notation does not only serve as a way to mark important mathematical ideas. In an inquiry-oriented classroom, the introduction of language and notation serves also as a way for teachers to support the use of a common way of thinking about a mathematical idea, which then serves as a launching point for the construction of more formal mathematics. Rasmussen and Marrongelle [26] indicate that inquiry-oriented instructors often notate student thinking in ways that allow the class to use that thinking to then solve new problems.

3.3.1. Description of developing a shared understanding in action

The teacher asks students to reflect on the contributions of other students, e.g., "How are you all making sense of what this group is starting to think about here?" The teacher encourages students to ask one another questions about alternate approaches and ways of thinking. Additionally, the instructor often points out ideas other students had while working on the task, as a way to support the students' progress. The teacher and students share responsibility for the development of the mathematical ideas.

When appropriate, the teacher records and notates student contributions with (formal) mathematical language and notation, as a way to foster a common vocabulary for the classroom community. The teacher listens to and explores students' contributions when appropriate, and utilizes these ideas to inform the lesson and follow-up questions that address any inconsistencies across the students' understanding of a particular idea. Questions about student contributions are asked to the class. The teacher uses productive intermediate student progress to redirect and focus small-group work towards the important mathematical ideas. The instructor often takes advantage of student thinking that he/she felt promoted the mathematical agenda by shifting focus to productive student-generated justification.

3.4. Connecting to standard mathematical language and notation

One of the major tenets of inquiry-oriented instruction is the idea that formal mathematics emerges from students' informal understandings [5]. As noted by Gravemeijer and Doorman [6]:

> there will always be a tension between a bottom-up approach that capitalizes on the inventions of the students and the need, (a) to reach certain given educational goals, and (b) to plan instructional activities in advance. As a consequence, a top-down element is inevitable in instruction [6, p. 9].

Conceptually speaking, one way for a teacher to approach this tension is to act as a broker "between the entire classroom community and the broader mathematical community by the insertion of formal convention and terminology" [29, p. 201]. Within the relevant research literature there exist two complementary ways by which teachers can help the students connect their own mathematical ways of reasoning to mathematically standard, and in some cases quite formal, language and notation: introducing language and notation as a way to record student thinking, and supporting the formalization of the students' ideas and contributions.

As described by Rasmussen et al. [29]:

> in contrast to more traditional teaching in which formal or conventional terminology is often the starting place for students' mathematical work, this teacher [one implementing an inquiry-oriented curriculum] chose to introduce the formal mathematical language only after the underlying idea had essentially been reinvented by the students [29, p. 203].

By using language as a way of capturing student thinking, as opposed to launching student thinking, the formal mathematics is treated as an extension of the students' understanding, instead of something external to the students' thinking. The introduction of (meaningful) language and notation is just one way in which teachers can support the formalization of student contributions and reasoning. Other ways are: translating informal, and sometimes incomplete, student contributions into testable conjectures [15]; introducing transformational records "that are initially used to record student thinking and that are later used by students to solve new problems" [28, p. 389]; providing generative alternatives intended to "foster particular social norms for explanation and that generate student justifications for the validity of these alternatives" [28, p. 389]. Each one of these options supports the teacher in carrying out a crucial role in inquiry-oriented instruction: "linking student-generated solution methods to disciplinary methods and important mathematical ideas" [11, p. 648].

3.4.1. Description of connecting to standard language and mathematical notation in action

The teacher notates student contributions with formal mathematical language and notation. This is done only after the students have had a chance to informally develop the mathematical concepts, and when it is likely that the students could make sense of the formal mathematics by connecting it to their previous work. The teacher explicitly prompts the students to make connections between their work and the more formal mathematical notation. For instance, the students are asked to translate and/or interpret the formal mathematical notation introduced by the teacher (which may not match the exact notation used by the students while completing the tasks), by utilizing their own generated ideas.

4. AN EXAMPLE FROM INQUIRY-ORIENTED LINEAR ALGEBRA

Over the last decade, a team of mathematics education researchers has been developing instructional materials for an inquiry-oriented course in introductory linear algebra [29, 35, 36, 37]. True to Realistic Mathematics Education, the concepts of span and linear independence grow out of the students' work in an experientially real task setting: in this case, the movements of a magic carpet and a hoverboard (for a full description see [37]). The first task in the inquiry-oriented linear algebra curriculum asks students to imagine using a hoverboard (with restricted movement) and a magic carpet (with restricted movement) to find Old Man Gauss given the coordinates of his house. In the subsequent task (see Figure 1), the students are asked if there are any locations he can hide that their modes of transportation cannot reach. This task sequence was designed to serve several purposes, including an introduction to the concepts of linear combination and span [37]. Here we provide a description of the implementation of the magic carpet task sequence by Dr. Patton (a mathematician who was not part of the design/research team that developed the inquiry-oriented linear algebra (IOLA) materials).

During Dr. Patton's implementation of this task sequence, she initiated the "Can Old Man Gauss Hide" task by asking the students to work in small groups to determine whether there was a place they would not be able to reach using the hoverboard and the magic carpet. During the small-group work, the students

You are a young traveler, leaving home for the first time. Your parents want to help you on your journey, so just before your departure, they give you two gifts. Specifically, they give you two forms of transportation: a hover board and a magic carpet. Your parents inform you that both the hover board and the magic carpet have restrictions in how they operate:

We denote the restriction on the *hover board's* movement by the vector $\begin{bmatrix}3\\1\end{bmatrix}$.
By this we mean that if the hover board traveled "forward" for one hour, it would move along a "diagonal" path that would result in a displacement of 3 units East and 1 unit North of its starting location.

We denote the restriction on the *magic carpet's* movement by the vector $\begin{bmatrix}1\\2\end{bmatrix}$.
By this we mean that if the magic carpet traveled "forward" for one hour, it would move along a "diagonal" path that would result in a displacement of 1 unit East and 2 units North of its starting location.

SCENARIO TWO: HIDE-AND-SEEK

Old Man Gauss wants to move to a cabin in a different location. You are not sure whether Gauss is just trying to test your wits at finding him or if he actually wants to hide somewhere that you can't visit him.

Are there some locations that he can hide and you cannot reach him with these two modes of transportation? Describe the places that you can reach using a combination of the hover board and the magic carpet and those you cannot. Specify these geometrically and algebraically. Include a symbolic representation using vector notation. Also, include a convincing argument supporting your answer.

Figure 1. The can Old Mann Gauss Hide task from IOLA.

worked together to investigate which points in the x-y plane could and could not be reached. Dr. Patton monitored the students' work and asked pressing questions in order to be able to better understand how the students understood the task (e.g., What does it mean to ride the hoverboard in reverse?) and to get insight into their solution strategies and reasoning. By doing so, Dr. Patton supported students in generating their own approaches and reasoning in relation to a mathematically meaningful task. By inquiring into the students' thinking, she was better able to support the students' inquiry into the mathematics. In addition, this allowed her to facilitate the generation of student ways of reasoning and solution paths without giving away answers.

To build on the students' thinking, Dr. Patton then asked students to present their group work, sequencing the presentations in a way that allowed her to facilitate a discussion about the similarities and differences in the various approaches. During this discussion, Dr. Patton made an explicit point to highlight student approaches and inscriptions that anticipated key mathematical ideas of the lesson. This sequence of presentations culminated in one group's argument that Gauss could not hide because for any position (x, y), they could algebraically solve for how long they needed to ride the hoverboard and magic carpet to find him (t_1 and t_2, respectively). Their work can be seen in Figure 2.

With the group's discovery of one of the key ideas of the lesson, Dr. Patton's role shifted toward supporting the development of a shared understanding throughout the entire class. She did this by asking the students to return to their groups and reflect on the contributions of other students. To further support this development, Dr. Patton directed the students to ideas other students expressed while working on the task, ideas that she deemed as important for the students' construction of the key mathematical ideas.

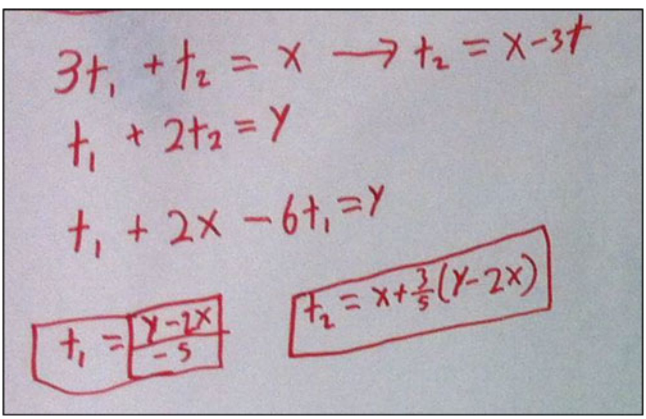

Figure 2. Student work from the Can Old Man Gauss Hide task.

Finally, Dr. Patton used the ideas the students discussed in the presentation of their approach as a way to introduce the ideas of linear combination and span. Explaining that:

> What we've done is express generic vector A, B [renaming and referring to the (*x, y*) from the students' work] as a linear combination of our hoverboard and magic carpet vectors.

In other words, Dr. Patton supported the formalization of the students' generated ideas and contributions in a way that connected the students' work to the mathematically standard language and notation of linear combination and span.

5. DISCUSSION

The teaching and learning of mathematics through inquiry has been steadily gaining traction in undergraduate mathematics education. However, the notion of inquiry, even just within the IBL community, is quite diverse. IBL has roots with the "Moore Method," but has grown to incorporate other perspectives on inquiry. For instance, Rasmussen and Kwon [27] highlight the importance of both student and teacher inquiry, whereas others are incorporating influences from scientific inquiry, with Katz and Thoren defining mathematical inquiry as the skills needed to "ask and explore new questions after they leave our classrooms" [15, p. 9]. These varied uses and notions around the word *inquiry* have had some important impacts on undergraduate mathematics education. On the one hand, "inquiry" as an umbrella term has facilitated the growth of a community of teachers who are interested in non-lecture instruction, providing a descriptor that can be utilized to help indicate the nature of the "non-lecture" classroom. On the other hand, communication around teaching practice and intent can be hindered by this catch-all term, as it does not provide a vernacular that allows for nuanced conversations about particular forms of "non-lecture."

Here our goal was to draw on our collective experience with inquiry-oriented instruction, situated within the theory of Realistic Mathematics Education, in order to present a vision of inquiry-oriented instruction framed around four instructional principles: generating student ways of reasoning, building on student contributions, developing a shared understanding, and connecting to standard mathematical language and notation. To better understand the nuances of these principles, and describe the role of each of these principles in instruction, we first looked at the inquiry-oriented and Realistic Mathematics Education theory and research literature. We then looked at the more extensive K-16 literature base to understand, and illustrate, how these principles can be enacted.

We hope that, by putting forward this characterization, we can help facilitate reflection and communication. In a lot of ways, teaching is a craft in which we incorporate values, goals, and philosophies into our daily practice, as opposed to a set of procedures and methods that we execute. As a result, the guiding principles we hold and the instructional practices we carry out can be rather implicit and situated. By providing this framing of inquiry-oriented instruction, we are offering a language that supports examining and reflecting on our own instruction, and also allows for meaningful communication with others about commonalities and differences in our instruction and our goals.

REFERENCES

1. Borko, H. 2004. Professional development and teacher learning: Mapping the terrain. *Educational Researcher*. 33(8): 3–15.
2. Brendehur, J. and J. Frykholm. 2000. Promoting mathematical communication in the classroom: Two preservice teachers' conceptions and practices. *Journal of Mathematics Teacher Education*. 3: 125–153.
3. Engle, R. A. and F. R. Conant. 2002. Guiding principles for fostering productive disciplinary engagement: Explaining an emergent argument in a community of learners classroom. *Cognition and Instruction*. 20(4): 399–483.
4. Freudenthal, H. 1973. *Mathematics as an Educational Task*. Dordrecht, The Netherlands: Reidel.
5. Gravemeijer, K. 1999. How emergent models may foster the constitution of formal mathematics. *Mathematical Thinking and Learning*. 1(2): 155–177.
6. Gravemeijer, K. and M. Doorman.1999. Context problems in realistic mathematics education: A calculus course as an example. *Educational Studies in Mathematics*. 39(1/3): 111–129.
7. Henningsen, M. and M. K. Stein. 1997. Mathematical tasks and student cognition: Classroom-based factors that support and inhibit high-level mathematical thinking and reasoning. *Journal for Research in Mathematics Education*. 28(5): 524–549.
8. Hiebert, J. 1997. *Making Sense: Teaching and Learning Mathematics with Understanding*. Portsmouth, NH: Heinemann.
9. Hiebert, J. and D. Wearne.1993. Instructional tasks, classroom discourse, and students' learning in second-grade arithmetic. *American Educational Research Journal*. 30(2): 393–425.
10. Hufferd-Ackles, K., K. C. Fuson, and M. G. Sherin. 2004. Describing levels and components of a math-talk learning community. *Journal for Research in Mathematics Education*. 35(2): 81–116.
11. Jackson, K., A. Garrison, J. Wilson, L. Gibbons, and E. Shahan. 2013. Exploring relationships between setting up complex tasks and opportunities to learn in concluding whole-class discussions in middle-grades

mathematics instruction. *Journal for Research in Mathematics Education*. 44: 646–682.
12. Johnson, E. 2013. Teachers' mathematical activity in inquiry-oriented instruction. *The Journal of Mathematical Behavior*. 32(4): 761–775.
13. Johnson, E., J. Caughman, J. Fredericks, and L. Gibson. 2013. Implementing inquiry-oriented curriculum: From the mathematicians' perspective. *The Journal of Mathematical Behavior*. 32(4): 743–760.
14. Johnson, E. M., and S. P. Larsen. 2012. Teacher listening: The role of knowledge of content and students. *The Journal of Mathematical Behavior*. 31(1): 117–129.
15. Katz, B. P. and E. Thoren. 2014. Call for papers for *PRIMUS* special issue on teaching inquiry. *PRIMUS*. 24: 9–10.
16. Keene, K. A. 2007. A characterization of dynamic reasoning: Reasoning with time as parameter. *The Journal of Mathematical Behavior*. 26(3): 230–246.
17. Kogan, M. and S. L. Laursen. 2014. Assessing long-term effects of inquiry-based learning: A case study from college mathematics. *Innovative Higher Education*. 39(3): 183–199.
18. Kuster, G., and E. Johnson. 2016. Inquiry oriented instruction: A conceptualization of the components and practices. Paper presented at the Nineteenth Annual Conference on Research in Undergraduate Mathematics Education, Pittsburgh, PA.
19. Larsen, S. P. 2013. A local instructional theory for the guided reinvention of the group and isomorphism concepts. *The Journal of Mathematical Behavior*. 32(4): 712–725.
20. Larson, C., and M. Zandieh. 2013. Three interpretations of the matrix equation Ax = b. *For the Learning of Mathematics*. 33(2): 11–17.
21. Larsen, S., E. Johnson, and K. Weber. (Eds.). 2013. The teaching abstract algebra for understanding project: Designing and scaling up a curriculum innovation. *Journal of Mathematical Behavior*. 32(4).
22. Laursen, S. L., M. L. Hassi, M. Kogan, and T. J. Weston. 2014. Benefits for women and men of inquiry-based learning in college mathematics: A multi-institution study. *Journal for Research in Mathematics Education*. 45(4): 406–418.
23. Leatham, K. R., B. E. Peterson, S. L. Stockero, and L. R. van Zoest. 2015. Conceptualizing mathematically significant pedagogical opportunities to build on student thinking. *Journal for Research in Mathematics Education*. 46(1): 88–124.
24. Munter, C. 2014. Developing visions of high-quality mathematics instruction. *Journal for Research in Mathematics Education*. 45(5): 584–635.
25. Rasmussen, C., O. Kwon, K. Allen, K. Marrongelle, and M. Burtch 2006. Capitalizing on advances in K-12 mathematics education in undergraduate mathematics: An Inquiry-oriented approach to differential equations. *Asia Pacific Education Review*. 7: 85–93.

26. Rasmussen, C., O. Kwon, and K. Marrongelle. 2008. A framework for interpreting inquiry-oriented teaching. Paper presented at The Eleventh Conference on Research in Undergraduate Mathematics Education, San Diego, CA.
27. Rasmussen, C. and O. N. Kwon. 2007. An inquiry-oriented approach to undergraduate mathematics. *The Journal of Mathematical Behavior*. 26(3): 189–194.
28. Rasmussen, C. and K. Marrongelle. 2006. Pedagogical content tools: Integrating student reasoning and mathematics in instruction. *Journal for Research in Mathematics Education*. 37(5): 388–420.
29. Rasmussen, C., M. Zandieh, and M. Wawro. 2009. How do you know which way the arrows go? The emergence and brokering of a classroom mathematics practice. In W.-M. Roth (Ed.), *Mathematical Representation at the Interface of Body and Culture*, pp. 171–218. Charlotte, NC: Information Age Publishing.
30. Sherin, M. G. 2002. A balancing act: Developing a discourse community in a mathematics classroom. *Journal of Mathematics Teacher Education*. 5(3): 205–233.
31. Speer, N. M. and J. F. Wagner. 2009. Knowledge needed by a teacher to provide analytic scaffolding during undergraduate mathematics classroom discussions. *Journal for Research in Mathematics Education*. 40(5): 530–562.
32. Stein, M. K., R. A. Engle, M. S. Smith, and E. K. Hughes. 2008. Orchestrating productive mathematical discussions: Five practices for helping teachers move beyond show and tell. *Mathematical Thinking and Learning*. 10(4): 313–340.
33. Stephan, M. and C. Rasmussen. 2002. Classroom mathematical practices in differential equations. *The Journal of Mathematical Behavior*. 21(4): 459–490.
34. Tabach, M., R. Hershkowitz, C. Rasmussen, and T. Dreyfus. 2014. Knowledge shifts in the classroom – A case study. *Journal of Mathematics Behavior*. 33: 192–208.
35. Wawro, M. 2015. Reasoning about solutions in linear algebra: The case of Abraham and the Invertible Matrix Theorem. *International Journal of Research in Undergraduate Mathematics Education*. 1(3): 315–338.
36. Wawro, M., C. Rasmussen, M. Zandieh, G. Sweeney, and C. Larson. 2012. An inquiry-oriented approach to span and linear independence: The case of the Magic Carpet Ride sequence. *PRIMUS*. 22(8): 577–599.
37. Wawro, M., C. Rasmussen, M. Zandieh, and C. Larson. 2013. Design research within undergraduate mathematics education: An example from introductory linear algebra. In T. Plomp, & N. Nieveen (Eds.), *Educational design research - Part B: Illustrative cases*, pp. 905–925. Enschede, the Netherlands: SLO.

38. Yackel, E., M. Stephan, C. Rasmussen, and D. Underwood. 2003. Didactising: Continuing the work of Leen Streefland. *Educational Studies in Mathematics*. 54(1): 101–126.

BIOGRAPHICAL SKETCHES

George Kuster is an assistant professor in the Department of Mathematics at Christopher Newport University. His research focuses on student thinking and learning in differential equations. The goals of this research are to understand how students' notions of function, rate of change, and solution are utilized and change as they construct an understanding of differential equations. In addition to this, he is also interested in the implementation of inquiry-oriented mathematics and its connections to student learning.

Estrella Johnson is an assistant professor of mathematics education in the Department of Mathematics at Virginia Tech. Her research focuses on the mathematical work done in the moment as mathematicians implement inquiry-oriented curricula. The goals of this research are to better understand the work carried out by teachers as they implement inquiry-oriented curriculum and to understand the ways in which teachers' mathematical work influences the learning of their students.

Karen Keene is an associate professor of mathematics education. She conducts research in undergraduate mathematics education, primarily concerning differential equations teaching and learning. Additionally she researches the social construction of mathematical meaning in undergraduate classrooms. Her second area of research is lies within secondary teacher education focusing on teachers' content knowledge and how it connects to their teaching and curriculum development.

Christine Andrews-Larson is an assistant professor of mathematics education in the College of Education at Florida State University. Her research focuses on teacher learning, with a particular emphasis on teacher workgroups and professional networks. She is currently working to coordinate research on student learning, and teacher professional development for the purpose of understanding how to scale up inquiry-oriented instruction with a focus on post-secondary mathematics.